Proof of Euler's Formula

Euler's Formula states that $e^{i\theta} = \cos\theta + i\sin\theta$.

Dividing both sides of the above equation by $e^{i\theta}$ should yield:

$$\frac{e^{i\theta}}{e^{i\theta}} = \frac{\cos\theta + i\sin\theta}{e^{i\theta}}$$

$$\Rightarrow 1 = \frac{\cos\theta + i\sin\theta}{e^{i\theta}} = (\cos\theta + i\sin\theta)e^{-i\theta}$$

Thus, Euler's Theorem states that $(\cos\theta + i\sin\theta)e^{-i\theta} = 1$ **always**. If we can prove that the relationship $(\cos\theta + i\sin\theta)e^{-i\theta}$ is always equal to 1, then Euler's Theorem must be true. In other words, we must prove that $(\cos\theta + i\sin\theta)e^{-i\theta}$ is a constant function which does not vary. Let $f(\theta) = (\cos\theta + i\sin\theta)e^{-i\theta}$. The derivative of a function measures how that function varies with respect to the variation of some argument. If a constant function does not vary, its derivative (where defined) will be zero.

$$\frac{d}{d\theta}f(\theta) = \frac{d}{d\theta}(\cos\theta + i\sin\theta)\cdot e^{-i\theta} + (\cos\theta + i\sin\theta)\cdot\frac{d}{d\theta}e^{-i\theta}$$

$$= (-\sin\theta + i\cos\theta)\cdot e^{-i\theta} + (\cos\theta + i\sin\theta)\cdot\left(-ie^{-i\theta}\right)$$

$$= (-\sin\theta + i\cos\theta)\cdot e^{-i\theta} - (i\cos\theta + i^2\sin\theta)\cdot\left(e^{-i\theta}\right)$$

$$= (-\sin\theta + i\cos\theta)\cdot e^{-i\theta} - (i\cos\theta - \sin\theta)\cdot\left(e^{-i\theta}\right)$$

$$= [(-\sin\theta + i\cos\theta) - (i\cos\theta - \sin\theta)]e^{-i\theta}$$

$$= [-\sin\theta + i\cos\theta - i\cos\theta + \sin\theta]e^{-i\theta} = 0$$

Thus, $f(\theta)$ is a constant function which does not vary. We have just proven that $(\cos\theta + i\sin\theta)e^{-i\theta}$ will always take on the same value. To find that value, we substitute any θ value, say $\theta = 0$, into $f(\theta)$. Thus, $f(0) = (\cos 0 + i\sin 0)e^{-i0} = 1$. We conclude that $(\cos\theta + i\sin\theta)e^{-i\theta} = 1$ **always**, thus proving Euler's Theorem.